Recovering Quivers from Derived Quiver Representations

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Recovering quivers from derived quiver representations

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Abstract

We compute Balmer's prime spectrum for the derived category of quiver representations for a finite ordered quiver with the vertex-wise tensor product and show that it does not recover the quiver. We then associate an algebra to every k-linear triangulated tensor category and show that the path algebra can be recovered in this way.

1 Introduction

1.1 Introduction

(1.1.1) For every essentially small tensor triangulated category T, Balmer [3] defined functorially a locally ringed space $\mathrm{Spec}(T)$, called its prime spectrum. It has been shown in some cases that the construction $T \mapsto \mathrm{Spec}(T)$ does not lose information in the sense that the category T can be reconstructed from $\mathrm{Spec}(T)$. For example, when $T_X = D(X)_{\mathrm{parf}}$ is the tensor triangulated category of perfect complexes on a scheme X [9, 3.1], there is a natural morphism

$$X \longrightarrow \mathrm{Spec}(T_X)$$

of locally ringed spaces. This comparison map is an isomorphism when X is quasi-compact and quasi-separated; see [2, Theorem 16 and Theorem 54].

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(1.1.2) Fix a field k. Let Q or (Q, R) be a quiver *with relations*; for legibility we will omit the relations R from the notation (Q, R) whenever convenient. Let $(Q, R)-\mathrm{Rep}$ be its abelian category of finite dimensional representations over k. Under suitable conditions on R (see (1.2.5)), this category is equipped with a natural *vertex-wise* tensor product which is an exact functor in each factor. Its bounded derived category $D(Q) := D^b((Q, R)-\mathrm{Rep})$ is then a tensor triangulated category when the relations satisfy (1.2.5). We will describe $\mathrm{Spec}(D(Q))$ in the case when Q is finite and ordered (that is, without non-trivial oriented cycles).

The ringed space $\mathrm{Spec}(D(Q))$ is not enough to recover Q, even in the case of quivers without relations; see (2.2.7). Moreover, $\mathrm{Spec}(D(Q))$ is not (yet) a quiver, hence we do not expect a comparison map

$$Q \xrightarrow{?} \mathrm{Spec}(D(Q)).$$

We show, however, that one can still recover Q (or, more accurately, its path algebra) from the tensor triangulated category $D(Q)$, at least in the case that Q is finite and ordered.

In general, it is possible for $D(Q)$ to be equivalent to $D(Q')$ where $Q \not\cong Q'$. For example, any two quivers of Dynkin type that have the same underlying graph have equivalent derived categories, by [6, Theorem 5.12]. More specifically,

$$D(\bullet \longrightarrow \bullet \longrightarrow \bullet) \simeq D(\bullet \longrightarrow \bullet \longleftarrow \bullet) \simeq D(\bullet \longleftarrow \bullet \longrightarrow \bullet).$$

It is striking, therefore, that the tensor structure on $D(Q)$ allows us to recover kQ.

(1.1.3) These results were motivated in part by the fact that the derived categories of some projective varieties are equivalent to derived categories of quiver representations. Consider for example the triangulated category $D(\mathbb{P}^m) = D^b(\mathrm{Coh}(\mathbb{P}^m))$. It is known to be equivalent to the category $D(S_m)$, where S_m is the quiver described in [4, Example 5.3 and Example 6.4]: it has $m+1$ ordered vertices and $m+1$ arrows between consecutive vertices, with commutativity relations (see (1.2.5)). Hence we have two different tensor products on the same triangulated category $T = D(\mathbb{P}^m) \cong D(S_m)$: one from the sheaf tensor product on $\mathbb{P}^m$ and the other the quiver tensor product. We will see that the two tensor products give very different spectra.

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1.2 Conventions and preliminary facts about quivers

(1.2.1) Fix a field k. Let $\mathbf{TT} = \mathbf{TT}_k$ denote the category of essentially small k-linear tensor triangulated categories, which are by definition triangulated categories T with a symmetric monoidal category structure $\otimes$ [8, XI.1] such that the functors $X \mapsto V \otimes X$ are exact for all $V \in T$.

Morphisms in this category are exact functors which are moreover strong symmetric monoidal functors [8, XI.2]; these will be called *tensor functors*.

We also denote the category Vect_k of finite dimensional k-vector spaces simply by Vect, and similarly $Q{-}\mathrm{Rep}_k$ by $Q{-}\mathrm{Rep}$ (this notation will be recalled in (1.2.3)).

(1.2.2) For any collection M of objects in a tensor triangulated category T, $\langle M \rangle$ denotes the smallest *tensor ideal* containing S. (For us, a tensor ideal is a proper, full, thick, triangulated subcategory J of T that satisfies $V \otimes W \in J$ for any $V \in J$ and $W \in T$.) A tensor ideal P is *prime* if $V \otimes W \in P$ implies that either $V \in P$ or $W \in P$.

(1.2.3) Here we recall some basic definitions and notations concerning quivers; for more details see [1, III.1].

A quiver $Q = (Q_0, Q_1)$ consists of two sets Q_0 (the set of vertices) and Q_1 (the set of arrows) along with two maps $s, t : Q_1 \to Q_0$ (source and target). A quiver is often realized as a directed graph, for example

$$\begin{tikzcd} 1 \arrow[r, "a"] \arrow[rr, bend right=60, "d"'] & 2 \arrow[r, bend left, "b"] \arrow[r, bend right, "c"'] & 3 \end{tikzcd}$$

represents a quiver Q with $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{a, b, c, d\}$ with the obvious definition of s and t.

Given a quiver Q, for each $v \in Q_0$ denote by e_v the trivial path; define $s(e_v) = t(e_v) = v$. A non-trivial path in Q is a sequence $p = a_1 a_2 a_3 \cdots a_\ell$ of arrows and trivial paths with $t(a_j) = s(a_{j+1})$); we extend the definition of s and t by setting $s(p) = s(a_1)$ and $t(p) = t(a_\ell)$. In the example above, $e_1 ab$ is a path from the vertex 1 to vertex 3.

Denote by kQ the k-algebra generated by the paths (including the trivial paths) with multiplication given by juxtaposition of paths, $pq = 0$ whenever $t(p) \neq s(q)$, and $e_{s(p)} p = p = p e_{t(p)}$. In the example above, we have $(e_1 a) \cdot (b) = e_1 ab$.

A quiver representation V of Q is a map that associates to every vertex $v \in Q_0$ an object V_v in Vect, and every arrow $a \in Q_1$ an element in $\mathrm{Hom}(V_{s(a)}, V_{t(a)})$; we set $V_{e_v} = \mathrm{Id}_{V_v}$. Then every path p in Q gives a morphism V_p from $V_{s(p)}$ to $V_{t(p)}$. By our convention of the product between paths, a path p acts *on the right* on vectors in $V_{s(p)}$.

Representations of Q naturally form an abelian category denoted by $Q-$Rep. The category $Q-$Rep is naturally identified with the category of kQ-modules (our convention is different from [1]: We identify $Q-$Rep with the category of *right* modules over the path algebra kQ).

The category $Q-$Rep has a natural symmetric associative vertex-wise tensor product given by

$$(V \otimes W)_* = V_* \otimes_k W_*,$$

where $*$ is either a vertex or a path in Q. It admits a unit object U given by $U_v = k$ and $U_p = \mathrm{Id}_k$ for every vertex v and path p. Thus $Q-$Rep is a symmetric monoidal category.

(1.2.4) We consider mostly *finite ordered quivers*, which means there are only finitely many vertices and arrows, and there are no oriented cycles. In this case we denote the vertices by integers $n = 1, 2, \ldots$, ordered in such a way that there are non-trivial paths from n to m only when $n < m$.

(1.2.5) We will also consider quivers with relations R. Any k-linear combination of paths in Q with the same source and target is called a relation. When we say Q (or sometimes (Q, R)) is a quiver with relations, R will always be a *two-sided ideal* of kQ generated by a set of relations. Denote by $(Q, R)-$Rep the full abelian subcategory of $Q-$Rep consisting of those representations V of Q such that

$$\sum_i \lambda_i V_{p_i} = 0$$

whenever $\sum \lambda_i p_i \in R$.

Consider the condition on the relations R of (Q, R) that the the subcategory $(Q, R)-$Rep is a monoidal subcategory of $Q-$Rep. More explicitly, we require that the unit object U in $Q-$Rep lies in the subcategory $(Q, R)-$Rep, and $(Q, R)-$Rep is closed under the tensor product.

Denote by $N \subseteq kQ$ the two-sided ideal generated by all the arrows. If R satisfies the condition above as well as $N^t \subseteq R \subseteq N^2$ for some $t \in \mathbb{N}$, then we say R is an ideal of *tensor relations*. The second condition in this definition allows us to decompose the k-algebra $\Lambda := kQ/R$ as in [1, page 59].

For example, relations generated by "commutativity relations" are tensor. This means that the ideal R is generated by relations of the form $p_i - q_i \in R$, where p_i and q_i are paths in Q with at least two arrows.

(1.2.6) For any quiver Q, a *full subquiver* Q' is a quiver with $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq Q_1$, that further inherits all arrows between vertices: that is, for any arrow $a \in Q_1$ between vertices $n, m \in Q'_0$, we have $a \in Q'_1$.

(1.2.7) Let Q be a quiver and $R \subseteq kQ$ an ideal of (not necessarily tensor) relations; let Q' be a full subquiver of Q. Denote by $f : Q' \to Q$ the inclusion map; it induces an injective k-linear map $kf : kQ' \to kQ$ between path algebras. The map kf is not a morphism of algebras since it does not send the identity to the identity, but it preserves path product.

Denote by $R \cap Q' \subseteq kQ'$ the subset of R consisting of elements all of whose paths are in Q'; more precisely, we let

$$R \cap Q' := (kf)^{-1}(R).$$

Then $R \cap Q'$ is an ideal of relations on Q'.

There is an exact *restriction functor*

$$f^* : Q\text{-Rep} \to Q'\text{-Rep}.$$

Then the image under f^* of the subcategory (Q, R)-Rep is contained in $(Q', R \cap Q')$-Rep, and we have an induced exact functor

$$f^* : (Q, R)\text{-Rep} \longrightarrow (Q', R \cap Q')\text{-Rep}.$$

We will need conditions under which the derived functor of f^* is full and essentially surjective. To this end it is convenient to find a right-inverse. Consider the exact *extension by zero* functor

$$f_* : Q'\text{-Rep} \to Q\text{-Rep}.$$

Observe that this functor respects the vertex-wise tensor product, but does not preserve the unit object in general.

Denote by $\bar{R} \subseteq kQ'$ the set of relations on Q' obtained by setting every path occurring in R but not completely in Q' to be zero. It is easy to see that $\bar{R}$ is an ideal: Indeed, it is the image of R under the natural surjection $kQ \to kQ'$ sending paths not in Q' to zero, and this surjection is an algebra homomorphism.

The image under f_* of $(Q', R \cap Q')$-Rep is contained in (Q, R)-Rep whenever $R \cap Q' = \bar{R}$. (Notice that the containment $\subseteq$ always holds.)

We say that R and Q' are *compatible* if the equality $R \cap Q' = \bar{R}$ holds (equivalently, $R \cap Q' \supseteq \bar{R}$), in which case we have

$$(Q, R)\text{-Rep} \underset{f_*}{\overset{f^*}{\rightleftarrows}} (Q', R')\text{-Rep},$$

where $R' := R \cap Q' = \bar{R}$. These functors satisfy $f^* f_* = \mathrm{Id}$ on (Q', R')-Rep.

Notice that if Q is ordered, R is generated by relations involving only non-trivial paths, and Q' consists of only one vertex, then R and Q' are always compatible, since in this case both $R \cap Q'$ and $\bar{R}$ are the zero ideal in $kQ' \cong k$.

(1.2.8) For example if Q is the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[dr, "c"'] & & 4, \\ & 3 \arrow[ur, "d"'] & \end{tikzcd}$$

R is the (two-sided) ideal generated by the relation $ab - cd$, and Q' is the full subquiver $1 \xrightarrow{a} 2 \xrightarrow{b} 4$, then $\bar{R}$ is generated by ab while $R \cap Q'$ is the zero ideal. Hence R and Q' in this example are not compatible.

(1.2.9) Let (Q, R) be a quiver with tensor relations, and Q' a full subquiver compatible with R; let $R' := R \cap Q' = \bar{R}$. We claim that R' is also an ideal of *tensor* relations (1.2.5). First, f^* sends the unit object U in $Q-$Rep to the unit object U' in $Q'-$Rep. By assumption $U \in (Q, R)-$Rep, and so $U' = f^*(U) \in (Q', R')-$Rep.

Next, let $V', W' \in (Q', R')-$Rep; we need to show that $V' \otimes W' \in (Q', R')-$Rep too. Since f^* respects tensor products, we have

$$V' \otimes W' = f^* f_*(V') \otimes f^* f_*(W') = f^*(f_*(V') \otimes f_*(W')).$$

But $f_*(V'), f_*(W') \in (Q, R)-$Rep, and so by assumption $f_*(V') \otimes f_*(W') \in (Q, R)-$Rep, whose image $V' \otimes W'$ under f^* then lies in $(Q', R')-$Rep, as required.

Finally, denote by $N' \subseteq kQ'$ the two-sided ideal generated by arrows in Q'. The containments $(N')^t \subseteq R' \subseteq (N')^2$ follow easily from $N^t \subseteq R \subseteq N^2$.

(1.2.10) Note that the full abelian subcategory $(Q, R)-$Rep of $Q-$Rep is in general not closed under extensions. In particular there is an inclusion functor $D(Q, R) \to D(Q)$, but $D(Q, R)$ mat not be a triangulated subcategory of $D(Q)$. This inclusion is an exact tensor functor (under the condition in (1.2.5)) that is injective on objects, but is not full in general: For example, consider the quiver Q with two vertices and two arrows a and b both going from one vertex to the other. Let R be generated by $a-b$. Then both simple objects in $Q-$Rep lie in $(Q,R)-$Rep, but most of their extensions do not lie in $(Q,R)-$Rep.

In general, all the simple objects of $Q-$Rep lie in $(Q,R)-$Rep, since all arrows act as the zero map in a simple representation.

(1.2.11) Let (Q,R) be a *finite ordered* quiver with tensor relations (1.2.5); let q be the number of its vertices. Denote its vertices by $1,2,\ldots,q$ so that there exists a non-trivial path from n to m only when $n<m$. (There is in general more than one such way to label the vertices.)

Let Q_ℓ be the full subquiver with vertices $\ell,\ell+1,\ldots,q$, for every $\ell\le q$.

(1.2.11.1) Lemma *Q_ℓ and R are compatible for every ℓ.*

Proof. With ℓ fixed, let $\bar{R}$ be as defined in (1.2.7), then we need to show $\bar{R}\subseteq R\cap Q_\ell$. For any relation $r\in R$ denote by $\bar{r}\in\bar{R}$ the relation obtained by setting every path not in Q_ℓ to be zero.

The ideal of relations R is generated by relations of the form $r=\sum p_j$ with each p_j a path between two fixed vertices, say from n to m; in particular every arrow occurring in r is between vertices v,w with $n\le v<w\le m$.

If $n\ge\ell$ then $r=\bar{r}$ is contained in Q_ℓ, since in this case Q_ℓ contains every arrow occurring in r.

If $n<\ell$ then at least one arrow in each path occurring in r does not lie in Q_ℓ, namely the arrow with source n. In this case $\bar{r}=0\in R\cap Q_\ell$ as well. □

With the ordering of the vertices fixed, we have a filtration $\cdots\subset K_{\ell+1}\subset K_\ell\subset K_{\ell-1}\subset\cdots U$ of the unit object $U\in Q$-Rep, so that the quotient $K_\ell/K_{\ell+1}$ is the simple object $U(\ell)$ supported at the vertex ℓ.

(1.2.11.2) Corollary *If (Q,R) is a finite ordered quiver with tensor relations, then every K_ℓ, $\ell=1,2,\ldots,q$, lies in the subcategory (Q,R)-Rep.*

Proof. Notice that if f denotes the inclusion of Q_ℓ in Q, then K_ℓ is the image under f_* of the unit object U_ℓ in Q_ℓ-Rep. Let $R_\ell:=R\cap Q_\ell$. Then by (1.2.11.1) and (1.2.9) we know that (Q_ℓ, R_ℓ) is a finite ordered quiver with tensor relations; in particular $U_\ell \in (Q_\ell, R_\ell)$-Rep. By (1.2.11.1) and (1.2.7) we have $K_\ell = f_* U_\ell \in (Q, R)$-Rep. $\square$

2 Balmer's constructions and their applications to quivers

In this section, we recall briefly the constructions in [3, Definition 2.1 and Definition 6.1], and apply them to quiver representations. We see that a great deal of information is lost.

2.1 Spectrum of a tensor triangulated category

(2.1.1) Let T be a tensor triangulated category. Here we recall Balmer's definition of the topological space $\operatorname{Spc} T$.

(2.1.1.1) Definition *The space* $\operatorname{Spc} T$ *is the set of prime* $\otimes$*-ideals of* T*. It is given the* Zariski topology*: a closed set is of the form*

$$\mathbf{Z}(S) := \{P \in \operatorname{Spc} T | S \cap P = \emptyset\},$$

where S is a set of objects in T.

(2.1.2) For example, Balmer has shown that for a (well-behaved) scheme X the topological space $\operatorname{Spc}(D(X)_{\text{parf}})$ is homeomorphic to X; see 2.2.3 for a precise statement.

(2.1.3) For any quiver (Q, R) with tensor relations (see (1.2.5)) let $D(Q, R)$, and often just $D(Q)$, be the bounded derived category of (Q, R)-Rep, equipped with the vertex-wise tensor product.

We now calculate $\operatorname{Spc} D(Q)$ for a finite ordered quiver Q with tensor relations, and show that the functor $Q \mapsto \operatorname{Spc} D(Q)$ retains merely the set Q_0 of vertices!

(2.1.4) Recall that every object V in $D(Q)$ is a bounded complex of objects in $Q-$Rep. In particular its cohomology $H(V) = \bigoplus H^i(V)$ is again an object in $Q-$Rep. We call the graded vector space $H(V)_n$ *the cohomology of V at the vertex n.*

(2.1.4.1) Lemma *Let Q be a finite ordered quiver with tensor relations. For any object V in $D(Q)$, we have $\langle V \rangle = \langle U(n) \,|\, H(V)_n \neq 0 \rangle$, where $U(n)$ is the simple representation corresponding to the vertex n.*

Proof. The unit object is given by the representation U with $U_n \cong k$ at each vertex n and identity maps between them for all arrows. Since Q is finite and ordered, U admits a filtration whose successive quotients are the simple representations $U(n)$; see (1.2.11).

Tensoring this filtration with V we see that $\langle V \rangle$ contains the object $V(n) = V \otimes U(n)$ with $V(n)_n = V_n$ and $V(n)_m = 0$ for $m \neq n$; all the arrows in the representation $V(n)$ are the zero map. On the other hand, V is an extension of the $V(n)$'s; thus $V \in \langle V(n) \rangle$. We have obtained that $\langle V \rangle = \langle V(n) \,|\, n = 1, 2, \ldots \rangle$.

The object $V(n) \in \langle V \rangle$ is a complex of vector spaces, and the construction in [5, Chapter III, § 1.4 Proposition] shows that it is isomorphic in $D(Q)$ to the complex $H(V)_n$ of vector spaces with zero differentials. Since $\langle V \rangle$ is a thick triangulated subcategory, it contains the simple representation $U(n)$ whenever $H(V)_n \neq 0$. $\square$

(2.1.4.2) Corollary *If V is a complex with non-zero cohomology at every vertex, then $\langle V \rangle = D(Q)$ is the unit ideal.* $\square$

(2.1.5) Consider the evaluation functor $V \mapsto V_n$ from $Q-$Rep to Vect. This functor is exact and preserves the tensor products as well as the unit object. Its derived functor from $D(Q)$ to $D^b(\mathrm{Vect}) \cong \oplus \mathrm{Vect}[j]$ then sends V to its cohomology $H(V)_n$ at n and is a tensor functor. The kernel of this derived functor is a tensor ideal, which we denote by P_n. It is the full subcategory of $D(Q)$ consisting of objects V with $H(V)_n = 0$.

(2.1.5.1) Theorem *Let Q be a finite ordered quiver with tensor relations. Then* $\operatorname{Spc} D(Q)$ *is the discrete space $\{P_n | \, n \in Q_0\}$.*

Proof. Clearly $P_n \supseteq \langle U(m) | m \neq n \rangle$. Suppose that $V \notin P_n$. Then $H(V)_n \neq 0$, hence $V \oplus \bigoplus_{m \neq n} U(m)$ is an object in $\langle P_n, V \rangle$ which has non-zero cohomology at every vertex. By (2.1.4.2) we have $\langle P_n, V \rangle = D(Q)$. Thus P_n is maximal, and by [3, Proposition 2.3(c)] P_n is prime.

Let I be an ideal that is not contained in any P_n. Then for every n we can find $V^n \in I$ with $H(V^n)_n \neq 0$. This implies that $\oplus V^n \in I$ has non-zero cohomology at every vertex, and I must then be the unit ideal. Thus the P_n are precisely the maximal ideals of $D(Q)$.

Let P be a prime ideal in $D(Q)$. By the previous paragraph it is contained in P_n for some n, in particular $U(n) \notin P$. But $U(m) \otimes U(n) = 0 \in P$ whenever $m \neq n$, hence $U(m) \in P$ for every $m \neq n$ since P is prime. That is, $P = P_n$. $\square$

(2.1.6) Let T be the triangulated category $D^b(\mathbb{P}^m)$ which by (1.1.3) is equivalent to $D^b(S_m)$. By [3, Corollary 5.6] the spectrum $\mathrm{Spc}(T, \otimes_{\mathbb{P}^m})$ under the sheaf tensor product is homeomorphic to $\mathbb{P}^m$, while by (2.1.5.1) above $\mathrm{Spc}(T, \otimes_{S_m})$ is $m+1$ discrete points. The same phenomenon happens for example for Grassmannians by [7], and more generally for varieties admitting a full strong exceptional set of objects.

2.2 The structure sheaf

(2.2.1) Let T be a tensor triangulated category. According to [3, Definition 2.1] the space $\mathrm{Spc}(T)$ comes with a map $\mathrm{supp}(-)$ from the objects of T to closed subsets of $\mathrm{Spc}(T)$ associating to every object V in T the set of prime ideals *not* containing V.

By comparing this with (2.1.5.1) we see that for any object V of $D(Q)$ we have

$$\mathrm{supp}(V) = \{P_n \mid H(V)_n \neq 0\}.$$

(2.2.2) Recall from [3, Definition 6.1] the definition of the structure sheaf on $\mathrm{Spc}(T)$: For any open subset $W \subseteq \mathrm{Spc}(T)$, denote by Z its complement, and T_Z the full triangulated subcategory of T consisting of objects a with $\mathrm{supp}(a)$ contained in Z. Then the localization functor [10, II.2.2.10] $T \to T/T_Z$ is a tensor functor between tensor triangulated categories. Denote still by U the image of the unit object $U \in T$ in T/T_Z, then $\mathcal{O}_{\mathrm{Spec}(T)}$ is by definition the sheafification of the presheaf of rings

$$W \mapsto \mathrm{End}_{T/T_Z}(U). \qquad (2.2.2.*)$$

The *spectrum* of a tensor triangulated category T is the pair $\mathrm{Spec}\, T := (\mathrm{Spc}\, T, \mathcal{O}_{\mathrm{Spec}(T)})$.

(2.2.3) If X is a quasi-compact and quasi-separated scheme, then $\mathrm{Spec}(D(X)_{\mathrm{parf}}) \cong X$ as locally ringed spaces [2, Theorem 54].

(2.2.4) We now consider the same construction for $T = D(Q)$.

(2.2.4.1) Theorem *Let Q be a finite ordered quiver with tensor relations. Then $\mathcal{O}_Q := \mathcal{O}_{\mathrm{Spec}(D(Q))}$ is the constant sheaf of algebras k. That is, for any $W \subseteq \mathrm{Spc}(D(Q)) = Q_0$, we have $\mathcal{O}_Q(W) \cong k^{\oplus W}$.*

Proof. Since $\mathrm{Spc}(D(Q))$ is a discrete topological space, it suffices to show that $\mathcal{O}_Q(\{v\}) \cong k$ on the open set $\{v\}$ consisting of one point. Let $Z := \mathrm{Spc}(D(Q)) - \{v\}$ be the complement of $\{v\}$ and Q' the full subquiver with only one vertex v. Denote by $f : Q' \to Q$ the inclusion. By identifying $D(Q')$ with $D^b(\mathrm{Vect})$ we see that f^* is exactly the functor $V \mapsto H(V)_v$ on $D(Q)$. In particular we have $T_Z = \ker(f^*)$ and an induced functor

$$\bar{f}^* : D(Q)/T_Z = D(Q)/\ker(f^*) \longrightarrow D(Q').$$

Since Q' is compatible with tensor relations (see (1.2.8)), we may apply the next proposition to conclude the proof. $\square$

(2.2.4.2) Proposition *Let (Q, R) be a finite ordered quiver with relations. Let $f : Q' \to Q$ be the inclusion of a full subquiver compatible with the relations; let $R' := R \cap Q'$ as in (1.2.7). Then the derived restriction functor $f^* : D(Q) \to D(Q')$ induces an equivalence*

$$\bar{f}^* : \overline{T} := D(Q)/\ker(f^*) \longrightarrow D(Q').$$

Proof. We claim that f^* is essentially surjective and full: Indeed, the derived functor of the extension by zero exact functor

$$f_* : Q'\text{--Rep} \to Q\text{--Rep}$$

is a right inverse of f^* on both objects and morphisms (see (1.2.7)). It follows that $\bar{f}^*$ is also essentially surjective and full, and so it remains to show that it is faithful.

Let $X, Y \in \mathrm{Ob}(D(Q)) = \mathrm{Ob}(\overline{T})$. An element of $\mathrm{Hom}_{\overline{T}}(X, Y)$ is represented by a diagram $X \xleftarrow{s} V \xrightarrow{g} Y$ in T, where s is such that f^*s is an isomorphism. This is thought of as a "morphism" $gs^{-1} : X \to Y$.

Now let $X \xleftarrow{s} V \xrightarrow{g} Y$ represent a morphism in $\overline{T}$ that maps to zero under $\bar{f}^*$. In particular $g : V \to Y$ is a morphism in $D(Q)$ such that $f^*g = 0$. By applying f^* to the distinguished triangle

$$V \xrightarrow{g} Y \xrightarrow{h} \mathrm{cone}(g)$$

we see that this implies f^*h has a left inverse $m : f^*\mathrm{cone}(g) \to f^*Y$ in $D(Q')$.

Let $\tilde{m} : \mathrm{cone}(g) \to Y$ be in $D(Q)$ such that $f^*\tilde{m}$ is equal to m; such an $\tilde{m}$ exists since f^* is full, as observed above.

Now $f^*(\tilde{m} \circ h) = m \circ f^*h$ is an isomorphism in $D(Q')$. Therefore $\mathrm{cone}(\tilde{m} \circ h)$ lies in $\ker(f^*)$ and this means that $\tilde{m} \circ h$ also becomes an isomorphism in $\overline{T}$. Hence g maps to zero in $\overline{T}$ since $(\tilde{m} \circ h) \circ g$ maps to zero in $\overline{T}$. So $X \stackrel{s}{\leftarrow} V \stackrel{g}{\to} Y$ represents the zero morphism in $\overline{T}$. $\square$

(2.2.5) The presheaf (2.2.2.$*$) formally contains more information than its sheafification $\mathcal{O}_{\mathrm{Spec}(T)}$. In the case of a quiver, this difference is significant. Let v, w be vertices in a quiver Q without relations, and let $W := \{v, w\}$ considered as an open subset of $\mathrm{Spec}(D(Q))$; denote by Q_W the full subquiver of Q consisting of vertices v and w.

Then $\mathrm{End}_{D(Q_W)}(U)$ is either k or $k \oplus k$, depending on whether there are arrows between v and w. Thus the presheaf (2.2.2.$*$) recovers the underlying graph of the quiver, while the structure sheaf recovers only the number of vertices.

(2.2.6) *Prime ideals in the path algebra:* Let Q be a finite ordered quiver with tensor relations. By [1, page 53] we know that the radical N of the path algebra kQ is nilpotent and is the ideal generated by all the non-trivial paths; further, kQ/N is isomorphic to the product of $\#Q_0$ copies of k.

By [1, Proposition III.1.6], the image $\bar{N}$ of N in the algebra $\Lambda := kQ/R$ is its radical. Since $R \subseteq N$, we have that $\Lambda/\bar{N} \cong kQ/N$ is also isomorphic to the product of $\#Q_0$ copies of k.

Any prime ideal of Λ contains $\bar{N}$, and so the prime ideals of Λ are naturally in bijection with prime ideals in $k^{\#Q_0}$: that is, with Q_0 and so with prime ideals in $D(Q) = D(Q, R)$. Therefore the global sections of $\mathcal{O}_{\mathrm{Spec}(D(Q))}$ are naturally isomorphic to $\Lambda/\bar{N}$. Further, since the map $\Lambda \to \Lambda/\bar{N}$ has a right inverse, the global sections of $\mathcal{O}_{\mathrm{Spec}(D(Q))}$ are isomorphic to a subalgebra of Λ.

(2.2.7) Let Q_i, $i \geq 1$, be the i-Kronecker quiver: the quiver (without relations) with two vertices and i arrows all going from one vertex to the other. There are obvious morphisms $Q_i \to Q_j$ with $i < j$, inducing functors $D(Q_j) \to D(Q_i)$ of tensor triangulated categories. These in turn induce morphisms

$$\mathrm{Spec}(D(Q_i)) \longrightarrow \mathrm{Spec}(D(Q_j))$$

of ringed spaces. From (2.1.5.1) we see easily that these are isomorphisms, in particular we cannot recover quivers from their prime spectra. Note that the presheaves (2.2.2.$*$) are isomorphic for all Q_i as well.

3 Functors of points

(3.1.1) Let $T, S \in \mathbf{TT}$. Define

$$T(S) := \mathrm{Hom}_{\mathbf{TT}}(T, S)/\cong,$$

where $\mathrm{Hom}_{\mathbf{TT}}(T, S)$ denotes the set of tensor functors from T to S (1.2.1), and $\cong$ denotes the equivalence relation of natural isomorphism.

Elements in $T(S)$ will be called *S-valued points in T*. We then have a set-valued covariant functor $T(-)$ on $\mathbf{TT}$ for every $T \in \mathbf{TT}$.

(3.1.2) If R is a commutative k-algebra, and S is the bounded derived category of finitely generated projective R-modules under the commutative tensor product, then we denote $T(S)$ also by $T(R)$. Elements in $T(R)$ will be called *R-rational points in T*.

For example, if $T = D^b(k)$, then $T(k)$ consists of just one point represented by the identity functor $D^b(k) \to D^b(k)$.

(3.1.3) Let Q be a finite ordered quiver with tensor relations (1.2.5); we will compute $D(Q)(k)$. We identify the tensor triangulated category $D^b(k) = D^b(\mathrm{Vect})$ with $\oplus\mathrm{Vect}[j]$ by taking cohomology of complexes of vector spaces.

The functors $V \mapsto H(V)_n$ in (2.1.5) are in $D(Q)(k)$, and conversely:

(3.1.3.1) Proposition *Let Q be a finite ordered quiver with tensor relations, then*

$$D(Q)(k) = \{(V \mapsto H(V)_n) \,|\, n \in Q_0\}.$$

Proof. Let $F : D(Q) \to D^b(k)$ be a tensor functor. Consider the filtration

$$\cdots \subset K_{n+1} \subset K_n \subset \cdots \subset U$$

of the unit object in $D(Q)$ such that the quotient K_n/K_{n+1} is the simple object $U(n)$ supported at the vertex n. By (1.2.11.2), each K_n lies in $D(Q) = D(Q, R)$.

Since F preserves the unit objects, $F(U)$ is isomorphic to k viewed as a graded vector space placed at degree 0. Hence on applying F to the distinguished triangle

$$K_{n+1} \xrightarrow{f_{n+1}} U \xrightarrow{g_{n+1}} U/K_{n+1},$$

we see that for any fixed vertex n exactly one of $F(f_{n+1})$ and $F(g_{n+1})$ is equal to 0, while the other one admits a one-sided inverse.

Note that for n sufficiently large we have $F(f_{n+1}) = 0$. Moreover, if $F(f_{n+1}) = 0$ then $F(f_{n+2}) = 0$: Indeed, suppose on the contrary that $F(f_{n+1}) = 0$ but $F(f_{n+2}) \neq 0$, then we get a contradiction by applying F to the following commutative diagram:

$$\begin{tikzcd} K_{n+2} \arrow[r, "f_{n+2}"] \arrow[d] & U \arrow[d, equal] \\ K_{n+1} \arrow[r, "f_{n+1}"] & U. \end{tikzcd}$$

So from now on let n be minimal so that $F(f_{n+1}) = 0$. Consider the following diagram of distinguished triangles, where by assumption $F(f_n)$ is non-zero and admits a right inverse m:

$$\begin{tikzcd} K_{n+1} \arrow[r, "f_{n+1}"] \arrow[d] & U \arrow[r] \arrow[d, equal] & U/K_{n+1} \arrow[d] \\ K_n \arrow[r, "f_n"] \arrow[d, "q"'] & U \arrow[r] \arrow[d] & U/K_n \arrow[d] \\ U(n) \arrow[r] & 0 \arrow[r] & U(n)[1] \end{tikzcd} \quad \overset{F}{\dashrightarrow} \quad \begin{tikzcd} F(K_{n+1}) \arrow[r, "0"] \arrow[d] & F(U) \arrow[r] \arrow[d, equal] & F(U/K_{n+1}) \arrow[d] \\ F(K_n) \arrow[r, "F(f_n)"] \arrow[d, "F(q)"'] & F(U) \arrow[r] \arrow[d] \arrow[l, bend left, "m"] & F(U/K_n) \arrow[d] \\ F(U(n)) \arrow[r] & 0 \arrow[r] & F(U(n))[1]. \end{tikzcd}$$

The map $F(q) \circ m : F(U) \to F(U(n))$ must be non-zero, since otherwise m lifts to a non-zero map from $F(U)$ to $F(K_{n+1})$, giving a right inverse to $F(f_{n+1})$, which is zero by assumption.

This says that $F(U(n))$ contains $F(U) = k$ as a direct summand, and when viewed as an object in $\oplus\mathrm{Vect}[j]$, $F(U(n))$ must be isomorphic to k since $U(n) \otimes U(n) = U(n)$.

The two functors $V \mapsto F(V) = F(U \otimes V)$ and $V \mapsto F(U(n) \otimes V)$ are then isomorphic:

$$F(V) \cong F(U \otimes V) \cong k \otimes F(V) \cong F(U(n)) \otimes F(V) \cong F(U(n) \otimes V).$$

Let $D(Q)_n$ denote the full subcategory of $D(Q)$ consisting of objects of the form $U(n) \otimes V$. The above then says that we have a factorization of the functor F:

$$\begin{tikzcd} D(Q) \arrow[rr, "F"] \arrow[dr, "U(n)\otimes -"'] & & D^b(k) \\ & D(Q)_n \arrow[ur, "F|_{D(Q)_n}"'] & \end{tikzcd}$$

The subcategory $D(Q)_n$ is closed under the tensor product on $D(Q)$, and is itself a symmetric monoidal category with $U(n)$ as the unit object. With this monoidal category structure it is monoidally equivalent to $D^b(k)$, and the functor $F|_{D(Q)_n}$ is strong monoidal, and so is an equivalence by the example in (3.1.2). Hence on identifying $D^b(k)$ with $\oplus \mathrm{Vect}[j]$ we see that the functor F is isomorphic to $V \mapsto H(V)_n$. $\square$

(3.1.4) To summarize, we have established bijections for any finite ordered quiver Q with tensor relations:

$$Q_0 \longrightarrow D(Q)(k) \longrightarrow \mathrm{Spc}(D(Q))$$

$$n \longmapsto (V \mapsto H(V)_n)$$

$$F \longmapsto \ker F.$$

(3.1.5) Let $F_n \in D(Q)(k)$ be the functor $V \mapsto H(V)_n$ corresponding to the vertex n. We now give a useful formula for the functors F_n, and a Yoneda lemma-type result.

Denote by M_n the indecomposable projective module of the k-algebra $\Lambda = kQ/R$ with simple quotient $U(n)$, namely, it is the *right* submodule of Λ generated by the trivial path e_n at the vertex n. That is, $M_n = e_n\Lambda$ is the k-vector space spanned by all paths starting at the vertex n; see [1, Page 59]. Then, as a right Λ-module,

$$\Lambda \cong \bigoplus_n M_n.$$

Under the usual identification of $(Q, R)-\mathrm{Rep}$ with $\mathrm{mod}-\Lambda$, we may view M_n as an object in the former abelian category.

(3.1.5.1) Lemma *Let Q be a finite ordered quiver with tensor relations.*

(i) For any vertex m, we have $F_m(\_\_) \cong R\mathrm{Hom}(M_m, \_\_)$ on $D(Q) = D(Q, R)$.

(ii) For any pair of vertices $n \leq m$, we have $\mathrm{Hom}(F_n, F_m) \cong F_m(M_n)$ *as k-vector spaces.*

Here $\mathrm{Hom}(F_n, F_m)$ denotes the k-vector space of all natural transformations.

Proof. (i) For any object V in $(Q, R)-\mathrm{Rep}$, let $\tilde{V}$ be the right Λ-module corresponding to it. Then we have $V_m = \tilde{V} e_m$.

Recall from (2.1.5) that F_m is the derived functor of the *exact* functor $V \mapsto V_m$ on $Q-\mathrm{Rep}$. Since $\mathrm{Hom}(M_m, \_\_)$ is also an exact functor, it suffices to show that

$$\mathrm{Hom}(M_m, V) \cong \mathrm{Hom}_\Lambda(M_m, \tilde{V})$$

is naturally isomorphic to V_m for $V \in (Q, R)-\mathrm{Rep}$. Hence we need to define a natural isomorphism

$$\mathrm{Hom}_\Lambda(M_m, \tilde{V}) \cong \tilde{V} e_m.$$

Consider the evaluation map from $\mathrm{Hom}_\Lambda(M_m, \tilde{V})$ to $\tilde{V}$, sending $f \mapsto f(e_m)$. It is an injection since $M_m = e_m \Lambda$ and f is a right Λ-module homomorphism.

The image of this evaluation map is exactly $\tilde{V} e_m$, since $f(e_m) = f(e_m^2) = f(e_m)e_m$, and for any $xe_m \in \tilde{V} e_m$, the assignment $f : e_m \mapsto xe_m$ extends to an element in $\mathrm{Hom}_\Lambda(M_m, \tilde{V})$.

(ii) This is a version of the Yoneda lemma. Since there are some subtleties pertaining to representable derived functors, we give an ad hoc proof.

We first define a map

$$\Phi : \mathrm{Hom}(F_n, F_m) \longrightarrow F_m(M_n).$$

So suppose ρ is a natural transformation from F_n to F_m, then ρ_{M_n} is a homomorphism from the one-dimensional vector space $F_n(M_n) \cong ke_n$ to $F_m(M_n)$. Let $\Phi(\rho) = \rho_{M_n}(e_n)$.

Now we define its inverse

$$\Psi : F_m(M_n) \longrightarrow \mathrm{Hom}(F_n, F_m).$$

Note that $F_m(M_n)$ is simply the k-vector space $e_n \Lambda e_m \subseteq \Lambda$. If $V \in D(Q)$ and a is in $F_m(M_n) = e_n \Lambda e_m$, then we let $\Psi(a)_V$ be the homomorphism from $F_n(V) = H(V)_n$ to $F_m(V) = H(V)_m$ given by right-multiplication by a.

Setting $V = M_n$ in the definition of Ψ above immediately gives $\Phi \circ \Psi = \mathrm{Id}$ on $F_m(M_n)$.

On the other hand, let $\rho \in \mathrm{Hom}(F_n, F_m)$. We need to show $\rho_V = (\Psi\Phi\rho)_V$ for every $V \in D(Q)$. First observe that it suffices to show this equality for objects V in $(Q,R)-$Rep: Indeed, if

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle in $D(Q)$ and a natural transformation between two exact functors on $D(Q)$ vanishes for any two of the three objects X, Y, Z, then it also vanishes for the third. Now every object in $D(Q)$ is a finite successive extension of objects in $(Q,R)-$Rep, and so by induction on the length of such extensions, it suffices to show that the difference natural transformation $\rho - \Psi\Phi\rho$ vanishes for every $V \in (Q,R)-$Rep.

So let V be a (Q,R)-representation, then we have a homomorphism

$$F_n(V) \xrightarrow{\rho_V} F_m(V).$$

Fix any vector $x \in F_n(V) = V_n$, the map $e_n \mapsto x$ extends to a right Λ-module morphism $e_n\Lambda = M_n \to \tilde{V}$. By naturality of ρ, we have a commutative diagram

$$\begin{tikzcd}
F_n(V) \arrow[r, "\rho_V"] & F_m(V) \\
F_n(M_n) \arrow[u] \arrow[r, "\rho_{M_n}"] & F_m(M_n). \arrow[u, "f"']
\end{tikzcd}$$

We make the identifications $F_m(M_n) = e_n\Lambda e_m$ and $\Phi(\rho) = \rho_{M_n}(e_n) \in e_n\Lambda e_m$. Then the diagram above sends vectors

$$\begin{tikzcd}
x \arrow[r, mapsto, "\rho_V"] & \rho_V(x) \\
e_n \arrow[u, mapsto] \arrow[r, mapsto, "\rho_{M_n}"] & \rho_{M_n}(e_n) = \Phi(\rho). \arrow[u, mapsto, "f"']
\end{tikzcd}$$

Since $\Phi(\rho) \in F_m(M_n) = e_n\Lambda e_m$, we have $\Phi(\rho) = e_n\Phi(\rho)$. Thus

$$\rho_V(x) = f(\Phi(\rho)) = f(e_n\Phi(\rho)) = f(e_n)\Phi(\rho) = x\Phi(\rho).$$

By definition of Ψ, this is $(\Psi\Phi\rho)_V(x)$. Thus $\rho_V = (\Psi\Phi\rho)_V$ as required. $\square$

4 Algebras associated to a tensor triangulated category

4.1 Defining the algebra

(4.1.1) Let $S, T \in \mathbf{TT}$. For any $F, G \in T(S)$ denote by $\mathrm{Hom}(F, G)$ the k-vector space of all natural transformations.

On the k-vector space

$$A(T, S) := \prod_{F,G \in T(S)} \mathrm{Hom}(F, G)$$

we then have a partially defined product by composition, which gives an associative, in general non-commutative algebra by defining the product between elements which are not composable to be zero. We are most interested in the special case:

$$A(T) := A(T, D^b(k)) := \prod_{F,G \in T(k)} \mathrm{Hom}(F, G).$$

(4.1.2) The functor $A(\_\_, \_\_)$ is contravariant in its first argument and covariant in its second.

4.2 Recovering finite ordered quivers

(4.2.1) We now show that a finite ordered quiver with tensor relations (1.2.5) can be recovered from its tensor triangulated category of representations. Recall that we denote by $D(Q) = D(Q, R)$ the bounded derived tensor category of finite dimensional (Q, R)-representations over the fixed field k.

(4.2.1.1) Theorem *Let Q be a finite ordered quiver with tensor relations. The k-algebra $\Lambda = kQ/R$ is naturally isomorphic to $A(D(Q))$. The isomorphism preserves the direct sum decompositions*

$$\Lambda \cong \bigoplus_{m,n \in Q_0} e_n \Lambda e_m \longrightarrow \bigoplus_{m,n \in Q_0} \mathrm{Hom}(F_n, F_m) \cong A(D(Q)).$$

(Recall that we have a bijection (see (3.1.4)) from the set of vertices Q_0 to $D(Q)(k)$ given by $n \mapsto (F_n : V \mapsto H(V)_n)$.)

Proof. We define a map $\phi : \Lambda \to A(D(Q))$ as follows: Let p be a path in Q from vertex n to vertex m, we need to define $\phi(p) \in \mathrm{Hom}(F_n, F_m)$. For any $V \in D(Q)$, $H(V)$ is a complex of (Q, R)-representations, and $H(V)(p)$ is a homomorphism from $H(V)_n = F_n(V)$ to $H(V)_m = F_m(V)$; so we define

$$\phi(p)_V := H(V)(p).$$

Since $H(V)$ is a Q-representation, the map ϕ gives a k-algebra homomorphism from kQ to $A(D(Q))$. Since $H(V)(r) = 0$ for every $r \in R$, it descends to a k-algebra homomorphism from Λ to $A(D(Q))$ which preserves the direct sum decomposition in the statement.

(Note that ϕ is in fact defined without assumptions on Q or R.)

Recall that $M_m = e_m\Lambda$ and $F_m(M_n) = e_n\Lambda e_m$. By (3.1.5.1) (with $V = M_n$) we have

$$\begin{tikzcd}
\mathrm{Hom}(F_n, F_m) \arrow[r, "\cong"] & F_m(M_n) \arrow[r, "\cong"] & \mathrm{Hom}_\Lambda(M_m, M_n) \\
\rho \arrow[r, mapsto] & \rho_{M_n}(e_n) & \\
& a \arrow[r, mapsto] & (e_m q \mapsto a e_m q).
\end{tikzcd}$$

We also have a natural isomorphism

$$\psi : \mathrm{Hom}_\Lambda(M_m, M_n) \longrightarrow e_n\Lambda e_m \subseteq e_n\Lambda = M_n,$$

sending $f : M_m \to M_n$ to $f(e_m) \in M_n$. It is now easy to show that ψ gives the inverse of ϕ on each summand:

$$\begin{tikzcd}
e_n\Lambda e_m \arrow[r, "\phi"] & \mathrm{Hom}(F_n, F_m) \arrow[r] & F_m(M_n) \arrow[r] & \mathrm{Hom}(M_m, M_n) \arrow[r, "\psi"] & e_n\Lambda e_m \\
p \arrow[r, mapsto] & (V \mapsto H(V)(p)) \arrow[r, mapsto] & \phi(p)_{M_n}(e_n) \arrow[d, equal] & & \\
& & e_n p \arrow[r, mapsto] & (e_m q \mapsto e_n p e_m q) \arrow[r, mapsto] & e_n p e_m = p.
\end{tikzcd}$$

The naturality in the statement means the following. Let (Q, R) and (Q', R') be two quivers with tensor relations; let $\Lambda := kQ/(R)$ and let $\Lambda' := kQ'/(R')$. Suppose there is a k-algebra homomorphism $\Lambda \to \Lambda'$. We obtain an induced restriction functor $D(Q', R') \to D(Q, R)$ and a homomorphism $A(D(Q, R)) \to A(D(Q', R'))$.

Then these homomorphisms form a commutative square with the isomorphisms $\Lambda \cong A(D(Q))$ and $\Lambda' \cong A(D(Q'))$ above. We leave the details to the reader. $\square$

Thus a finite, ordered quiver Q with tensor relations (or at least its path algebra) can be recovered from its tensor triangulated category $D(Q)$ of representations.

By [1, Theorem III.1.9(c)(d)], the quiver Q can be recovered from its path algebra if the ideal R of relations lies between N^2 and N^t for some integer t, where N is the radical of kQ.

4.3 Comparing the two constructions

(4.3.1) For a tensor triangulated category T, the structure sheaf $\mathcal{O}_{\operatorname{Spec} T}$ and the algebra $A(T)$ are naturally related. This follows from:

(4.3.1.1) Proposition *Let T, S be tensor triangulated categories, and let U be the unit object of T. Then $A(T,S)$ is naturally an $\operatorname{End}_T(U)$-algebra. More precisely, there is a natural map from $\operatorname{End}_T(U)$ to the center of $A(T,S)$.*

Proof. Let $g \in \operatorname{End}_T(U)$. Then it induces a natural transformation $\tilde{g} : \operatorname{Id}_T \to \operatorname{Id}_T$, where $\tilde{g}_V : V \to V$ for $V \in T$ is given by

$$V \xrightarrow{\cong} V \otimes U \xrightarrow{V \otimes g} V \otimes U \xrightarrow{\cong} V, \qquad (4.3.1.*)$$

where the isomorphisms are the compatibility isomorphisms of the unit object U.

Applying any tensor functor $F : T \to S$ to the diagram above gives a morphism

$$F(g)_V : F(V) \to F(V)$$

in S. Then $V \mapsto F(g)_V$ is a natural transformation from F to F since F sends the compatibility isomorphisms in T to those in S [8, XI.2].

If $g, h \in \operatorname{End}_T(U)$, then $V \otimes (gh) = (V \otimes g)(V \otimes h)$ and so $F(gh)_V = (F(g)_V)(F(h)_V)$ since $V \otimes -$ and $F(\_\_)$ are functors. Moreover, these functors are k-linear hence the map $g \mapsto F(g)$ from $\operatorname{End}_T(U)$ to $\operatorname{Hom}(F,F) = \operatorname{End}(F)$ is a k-algebra homomorphism: it sends the identity Id_U to the identity Id_F since the two isomorphisms in (4.3.1.$*$) are inverse to each other.

Hence we have defined a map $z : \operatorname{End}_T(U) \to A(T,S)$:

$$\operatorname{End}_T(U) \longrightarrow \prod_{F \in T(S)} \operatorname{Hom}(F,F) \hookrightarrow \prod_{F,G \in T(S)} \operatorname{Hom}(F,G) = A(T,S),$$

where the first arrow sends

$$g \mapsto z(g) := \prod_F F(g),$$

and the second arrow is the inclusion of the diagonal. Thus z is in fact a k-algebra homomorphism.

Finally, let $g \in \mathrm{End}_T(U)$, let $G, H \in T(S)$, and let $\beta \in \mathrm{Hom}(G,H)$. Then we need to show that

$$z(g)\beta = \beta z(g)$$

in $A(T,S)$. The definition of the multiplication in $A(T,S)$ gives

$$z(g)\beta = \left(\prod_F F(g)\right)\beta = H(g)\beta \in \mathrm{Hom}(G,H)$$

since $F(g)\beta = 0$ whenever $F \neq H$; similarly we have

$$\beta z(g) = \beta G(g) \in \mathrm{Hom}(G,H).$$

Let $V \in T$. Then by the naturality of β, the diagram

$$\begin{tikzcd} G(V) \arrow[r, "G(g)_V"] \arrow[d, "\beta_V"'] & G(V) \arrow[d, "\beta_V"] \\ H(V) \arrow[r, "H(g)_V"'] & H(V) \end{tikzcd}$$

commutes, since the two horizontal arrows are obtained by applying G and H respectively to (4.3.1.$*$). That is, $H(g)\beta = \beta G(g)$ in $A(T,S)$, and so we conclude that $z(g) \in Z(A(T,S))$.

The naturality in the statement means the following: Let $S \in \mathbf{TT}$ be fixed. If $z_T : \mathrm{End}_T(U) \to A(T,S)$ and $z_{T'} : \mathrm{End}_{T'}(U') \to A(T',S)$ are the k-algebra homomorphisms as defined above, and $K : T \to T'$ is a tensor functor, then the induced diagram

$$\begin{tikzcd} \mathrm{End}_T(U) \arrow[r, "z_T"] \arrow[d, "K(-)"'] & A(T,S) \\ \mathrm{End}_{T'}(U') \arrow[r, "z_{T'}"] & A(T',S) \arrow[u] \end{tikzcd}$$

is commutative. The verification of this is straightforward and we leave it to the reader. $\square$

(4.3.2) From the previous result, we see that for any tensor triangulated category T, there are ring homomorphisms

$$\begin{tikzcd} \mathrm{End}_T(U) \arrow[r, "\alpha"] \arrow[dr, "z"'] & \Gamma(\mathcal{O}_{\mathrm{Spec}\, T}) \\ & A(T), \end{tikzcd}$$

where α is induced from the canonical map from a presheaf to the associated sheaf. In general, there does not seem to be any reason why there should be a vertical map (in either direction) completing the triangle. However, we have:

(4.3.2.1) Proposition *Let Q be a finite ordered quiver with tensor relations and $T = D(Q) = D(Q, R)$. Then there are vertical maps so that the diagram*

$$\begin{tikzcd} \mathrm{End}_T(U) \arrow[r, "\alpha"] \arrow[dr, "z"'] & \Gamma(\mathcal{O}_{\mathrm{Spec}\, T}) \arrow[d, shift left] \\ & A(T) \arrow[u, shift left] \end{tikzcd}$$

commutes; further $\mathrm{End}_T(U) \cong Z(A(T))$.

Proof. It is easy to see that $\mathrm{End}_T(U) \cong k^{\pi_0(Q)}$ is the center of the algebra $\Lambda = kQ/R$; here π_0 denotes the set of equivalence classes of vertices in Q under the equivalence relation generated by the existence of arrows between them.

The vertical maps come from (2.2.6), using the isomorphism $A(T) \cong \Lambda$ (4.2.1.1). $\square$

(4.3.3) We note that for tensor triangulated categories $T = D^b(X)$ induced from a scheme X, the algebra $A(T)$ can be quite unpleasant, and does not necessarily recover X. For example, let k be an algebraically closed field, and let $T := D^b(\mathbb{P}^1_k)$. Then $T(k) = \mathbb{P}^1(k)$; that is, the only tensor functors from T to $D^b(k)$ are given by restriction to a k-point. The algebra $A(T)$ is then easily seen to be

$$A(T) = \prod_{p \in \mathbb{P}^1(k)} k(p),$$

the direct product of residue fields $k(p) \cong k$. Indeed, it suffices to show that if x, y are in $\mathbb{P}^1(k)$, then there are no non-zero natural transformations between the corresponding functors x^* and y^*.

Suppose on the contrary that ϕ is a non-zero natural transformation from x^* to y^*. Then there is a coherent sheaf $\mathcal{F}$ such that both fibres $x^*\mathcal{F}$ and $y^*\mathcal{F}$ are non-zero, and $\phi_{\mathcal{F}}$ is a non-zero map between them. In fact since $D^b(\mathbb{P}^1)$ is generated by $\mathcal{O}$ and $\mathcal{O}(1)$, we may assume that $\mathcal{F}$ is a line bundle. In this case the non-zero map $\phi_{\mathcal{F}}$ must be an isomorphism between the fibres $x^*\mathcal{F}$ and $y^*\mathcal{F}$.

Now let $\mathcal{F}' := y_* y^* \mathcal{F}$. Then we have $x^*\mathcal{F}' = 0$ but $y^*\mathcal{F} \to y^*\mathcal{F}' = y^* y_* y^* \mathcal{F}$ is a non-zero map. Hence we obtain a commutative diagram, by the naturality of ϕ:

$$\begin{tikzcd} x^*\mathcal{F} \arrow[r] \arrow[d, "\phi_{\mathcal{F}}"', "\cong"] & x^*\mathcal{F}' = 0 \arrow[d, "\phi_{\mathcal{F}'}"'] \\ y^*\mathcal{F} \arrow[r, "\neq 0"] & y^*\mathcal{F}'. \end{tikzcd}$$

But this implies that $\phi_{\mathcal{F}} = 0$, a contradiction.

Thus for triangulated tensor categories coming from algebraic geometry, Balmer's construction is much better than ours. It would be interesting to find a functorial construction that combines the good features of both and to prove a reconstruction theorem that generalizes simultaneously (4.2.1.1) and [3].

References

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